

THE 1-LINE OF THE K -THEORY BOUSFIELD-KAN SPECTRAL SEQUENCE FOR $Spin(2n+1)$

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1. STATEMENT OF RESULTS

The p -primary v_1 -periodic homotopy groups, $v_1^{-1}\pi_*(X;p)$, of a topological space X , as defined in [13], are a localization of the portion of the actual homotopy groups of X detected by K -theory. In [10, 4, 6], $v_1^{-1}\pi_*(X;p)$ was calculated for classical groups X and primes p in all cases except $(SO(n), 2)$. In this paper we make a first step toward the calculation of $v_1^{-1}\pi_*(SO(n); 2)$, which is of course isomorphic to $v_1^{-1}\pi_*(Spin(n); 2)$.

In [7], a Bousfield-Kan-type spectral sequence $E_r^{s,t}(X)$ based on periodic K -theory was introduced. It converges to the homotopy groups of the K -completion X_K^\wedge of the space X . In this paper, we will deal exclusively with the localization of this spectral sequence at the prime 2. We expect to prove in a subsequent paper that if $X = Spin(n)$ then $\pi_i(X_K^\wedge) \approx v_1^{-1}\pi_i(X; 2)$ for sufficiently large values of i . Thus, since v_1 -periodic homotopy groups are periodic, a computation of this spectral sequence for $X = Spin(n)$ and $t - s$ large would yield a complete computation of $v_1^{-1}\pi_*(Spin(n))$.

If X is an H -space such that $K_*(X)$ is a free commutative algebra on odd-dimensional classes (e.g. $X = Spin(n)$), then ([7, 4.9]) $E_2(X)$ is the homology of an unstable co-bar complex determined by these classes and their K_*K -coaction. The main result of this paper is an explicit computation of the 1-line $E_2^{1,t}(Spin(2n+1))$ of this spectral sequence when $p = 2$ and $X = Spin(2n+1)$, .

The first step toward this computation is the following general result, which was inspired by the results of Bousfield ([9]) at the odd primes. This result will be proved

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in Section 2. In it, $(-)^{\#}$ denotes the Pontryagin dual, and $PK^1(X)$ the primitives in $K^1(X)$.

Theorem 1.1. *If X is a simply-connected finite H -space with K_*X an exterior algebra on odd-dimensional classes of degree $\leq 2M + 1$, then $E_2^{1,2m}(X) = 0$ for all m , and, if $m > M$, then*

$$E_2^{1,2m+1}(X) \approx (PK^1(X)/\text{im}(\psi^r - r^m : r = 2, 3, -1))^{\#}.$$

For simplicity of notation, we define

$$\tilde{v}^{2m}(X) := PK^1(X)/\text{im}(\psi^r - r^m : r = 2, 3, -1)$$

and the closely-related functor

$$v^{2m}(X) := PK^1(X)/\text{im}(\psi^2, \psi^3 - 3^m, \psi^{-1} - (-1)^m).$$

The advantage of $\tilde{v}^*(-)$ is that it is more closely related to $E_2^{1,*+1}(-)$, while the advantage of $v^*(-)$ is that it is periodic and has simpler formulas. They are related by the following result, which will be proved at the end of this section.

Proposition 1.2. *If X is a simply-connected finite H -space with K_*X an exterior algebra on classes of degrees $2d_1 + 1, \dots, 2d_r + 1$, then $v^{2m}(X) \approx \tilde{v}^{2m}(X)$ if $m > \sum d_i$.*

The second purpose of this note is to compute $v^{2m}(Spin(2n + 1))$, which, by Theorem 1.1 and Proposition 1.2, is isomorphic to $E_2^{1,2m+1}(Spin(2n + 1))$ if $m > n^2$. Here we use the rational equivalence of $Spin(2n + 1)$ with $\prod_{i=1}^n S^{4i-1}$, and that the sum of the odd integers up to $2n - 1$ is n^2 .

We will show in Proposition 3.1 that there is an injection of Adams-modules

$$PK^1(Sp(n)) \rightarrow PK^1(Spin(2n + 1)),$$

and hence a morphism

$$v^{2m}(Sp(n)) \rightarrow v^{2m}(Spin(2n + 1)), \quad (1.3)$$

which is dual to a morphism of $E_2^{1,2m+1}(-)$. The following result was proved in [5]. Here and throughout, $\nu(-)$ denotes the exponent of 2 in a number.

Theorem 1.4. ([5]) *If m is odd and $m \geq 2n$, then $E_2^{1,2m+1}(Sp(n)) \approx \mathbf{Z}/2^{eSp(m,n)}$, where*

$$eSp(m, n) := \min(\nu(S_{m,j}) : j > 2n),$$

with $S_{m,j} = \sum (-1)^k \binom{j}{k} k^m$. If m is even, then $E_2^{1,2m+1}(Sp(n)) = \mathbf{Z}/2$.

Actually, the result in [5] was proved for the BP -based unstable Novikov spectral sequence, but one might use the change of rings theorem in [7], or mimic the calculation in [5] to yield the result for the K -based spectral sequence.

Our second main result is as follows. In light of 1.1, it gives $E_2^{1,2m+1}(Spin(2n+1))$ when $m > n^2$.

Theorem 1.5. *If $n = 3$ or $n \geq 5$, and m is odd, then*

$$v^{2m}(Spin(2n+1)) \approx \mathbf{Z}/2^{\min(eSp(m,n), \nu(R_1(m,n)), \nu(R_2(m,n)))} \oplus \mathbf{Z}/2^{\min(2+\nu(m+1), n)},$$

where

$$R_1(m, n) = \sum_{\text{odd } k \geq 1} k^m \left(\sum_{i=0}^{n-k} \binom{2n+1}{i} - 2 \sum_{t \geq 0} \binom{2n+2}{n-1-k-4t} \right)$$

and $R_2(m, n) =$

$$\left((2^{2n+1} - 3^{m+1} + 1) \sum_{\text{odd } k \geq 1} k^m \sum_{t \geq 0} \binom{2n+2}{n-1-k-4t} - 3 \cdot 2^{2n} \sum_{\text{odd } k \geq 1} k^m \sum_{t \geq 0} \binom{2n+1}{n-1-k-3t} \right) / 2^{\min(2+\nu(m+1), n)}.$$

If m is even, then $v^{2m}(Spin(2n+1)) \approx \mathbf{Z}/2 \oplus \mathbf{Z}/2$.

The sums over odd k could include even values as well, as long as $m > n^2$, since this is greater than the largest possible exponent of $v^*(Spin(2n+1))$. We will discuss the slight anomaly when $n = 4$ in Proposition 4.21. Since $Spin(5) \approx Sp(2)$, the case $n = 2$ has been covered in [6] (and is quite different).

The morphism (1.3) sends a generator to $g_1 + ag_2$, for some integer a . We could certainly determine a , but shall not bother to do so here.

We illustrate with concrete calculations when $n = 5$ and 6. These results are obtained by computer calculation, although we shall show after (4.20) how to obtain some of them by hand.

Proposition 1.6. *For $n = 5$ and 6, $v^{2m}(Spin(2n+1)) \approx \mathbf{Z}^{e_1(2n+1)} \oplus \mathbf{Z}^{e_2(2n+1)}$, where $e_1(2n+1)$ and $e_2(2n+1)$ are as in the following table, which also presents $eSp(m, n)$ for comparison.*

| n | m | $eSp(m, n)$ | $e_1(2n+1)$ | $e_2(2n+1)$ |
|-----|-----------|---------------------------|----------------------------|-------------|
| 5 | 3 mod 8 | 8 | 5 | 4 |
| 5 | 7 mod 8 | $\min(11, \nu(m-7) + 6)$ | $\min(11, \nu(m-103) + 2)$ | 5 |
| 5 | 1 mod 4 | $\min(14, \nu(m-73) + 6)$ | $\min(12, \nu(m-73) + 4)$ | 3 |
| 6 | 15 mod 16 | 11 | 6 | 6 |
| 6 | 7 mod 16 | 11 | $\min(9, \nu(m-23) + 4)$ | 5 |
| 6 | 3 mod 8 | $\min(15, \nu(m-75) + 9)$ | $\min(15, \nu(m-523) + 5)$ | 4 |
| 6 | 1 mod 4 | $\min(14, \nu(m-9) + 8)$ | $\min(13, \nu(m-201) + 5)$ | 3 |

Proof of Proposition 1.2. There is a rational equivalence $X \simeq \coprod S^{2d_i+1}$. Thus there are elements $x_i \in PK^1(X)$ for which $\psi^k(x_i) = k^{d_i}x_i$ for all k . Similarly to the procedure in [11], we can find a basis $\{w_1, \dots, w_r\}$ of $PK^1(X)_{(2)}$ such that each w_i is a rational linear combination of $\{x_j : j \geq i\}$, and hence the matrix of each ψ^k is triangular with $\{k^{d_1}, \dots, k^{d_r}\}$ along the diagonal. Thus the Adams module $PK^1(X)$ can be built from $PK^1(S^{2d_1+1}), \dots, PK^1(S^{2d_r+1})$ by short exact sequences, and by the Snake Lemma this implies the same of $v^{2m}(X)$ (built from $v^{2m}(S^{2d_i+1})$). Since $2^{d_i}v^{2m}(S^{2d_i+1}) = 0$ by [7], we deduce $2^{\sum d_i}v^{2m}(X) = 0$ (a very conservative estimate). If $m \geq \sum d_i$, then $\cdot 2^m$ is 0 in $v^{2m}(X)$. Since the only difference between $v^{2m}(X)$ and $\tilde{v}^{2m}(X)$ is multiples of 2^m , we conclude that they are equal. ■

2. PROOF OF THEOREM 1.1

In this section, we prove Theorem 1.1. We begin by observing that the exact sequence in E_2 induced by the coefficient sequence

$$0 \rightarrow \mathbf{Z} \rightarrow \mathbf{Q} \rightarrow \mathbf{Q}/\mathbf{Z} \rightarrow 0$$

induces isomorphisms $E_2^{s,t}(X; \mathbf{Q}/\mathbf{Z}) \rightarrow E_2^{s+1,t}(X)$ except when $s = 0, t = 2d_i + 1$. Hence there is an isomorphism when $s = 0, t > 2d_r + 1$. Everything is localized at 2. We have

$$E_2^{0,2m+1}(X; \mathbf{Q}/\mathbf{Z}) \approx \ker(\bar{d} : K_{2m+1}(X) \otimes \mathbf{Q}/\mathbf{Z} \rightarrow (K_*K \otimes K_*X \otimes \mathbf{Q})/U(X)), \quad (2.1)$$

where \bar{d} is induced from

$$d : (K_*(X) \otimes \mathbf{Q}, K_*(X)) \rightarrow (K_*K \otimes K_*X \otimes \mathbf{Q}, U(X)),$$

the boundary in the unstable cobar complex described in [7, §4]. The following result, whose proof is deferred until the end of this section, is crucial. Here $QK_*(-)$ denotes the indecomposables.

Proposition 2.2. *Suppose $QK_*(X)$ has basis $\{y_1, \dots, y_r\}$ as a K_* -module with $y_j \in K_{2d_j+1}(X)$, and $x \in K_*(X) \otimes \mathbf{Q}$ has $d(x) = \sum s_j \otimes y_j$, with $s_j \in K_*K \otimes \mathbf{Q}$. Then $x \in \ker(\bar{d})$ (of (2.1)) if and only if for all positive integers k and all subscripts j*

$$k^{d_j} \langle \psi^k, s_j \rangle \in K_{*(2)}. \quad (2.3)$$

The condition (2.3) says that each $k^{d_j} \langle \psi^k, s_j \rangle$ is a 2-local integer times the generator of $K_{|s_j|}$.

Now we prove Theorem 1.1. Suppose $\{y_1, \dots, y_r\}$ is as in Proposition 2.2 and

$$\psi : K_*X \rightarrow K_*K \otimes K_*X$$

satisfies

$$\psi(y_j) = \sum_i s_{i,j} \otimes y_i. \quad (2.4)$$

Let $u \in K_2$ denote the Bott class, so that $K_* = \mathbf{Z}_{(2)}[u]$, and let $v = \eta_R(u)$. Then

$$d(u^e y_j) = (v^e - u^e) \otimes y_j - \sum_{i \neq j} u^e s_{i,j} \otimes y_i.$$

Since, by [8, p.676], $\langle \psi^k, v \rangle = ku$ and $\langle \psi^k, u \rangle = u$, the expression whose coefficients correspond to $\langle \psi^k, s_j \rangle$ in (2.3) equals

$$k^e u^e y_j - \sum_{\text{all } i} \langle \psi^k, s_{i,j} \rangle u^e y_i.$$

A basis for $QK_{2m+1}(X)$ is given by $\{u^{m-d_1}y_1, \dots, u^{m-d_r}y_r\}$, and, by Proposition 2.2, $E_2^{0,2m+1}(X; \mathbf{Q}/\mathbf{Z})$ is the intersection, over all integers $k > 1$, of the kernel of the morphism

$$\delta_k : K_{2m+1}(X) \otimes \mathbf{Q}/\mathbf{Z} \rightarrow K_{2m+1}(X) \otimes \mathbf{Q}/\mathbf{Z}$$

defined by

$$\delta_k(u^{m-d_j}y_j) = -k^{m-d_j}u^{m-d_j}k^{d_j}y_j + u^{m-d_j} \sum_i \langle \psi^k, s_{i,j} \rangle k^{d_i}y_i.$$

Here we have negated the differential d for later convenience. Define $\alpha_{i,j,k} \in \mathbf{Z}$ by $\langle \psi^k, s_{i,j} \rangle k^{d_i} = \alpha_{i,j,k} u^{d_j - d_i}$, and define an $r \times r$ -matrix $A_k = (\alpha_{i,j,k})$. Then δ_k is a linear transformation of free \mathbf{Q}/\mathbf{Z} -modules with matrix $A_k - k^m I$.

Now let $w_i \in K^{2d_i+1}(X)$ be dual to y_i , and let $z_i = u^{d_i} w_i \in K^1(X)$. By [8, 11.19] applied to (2.4), we have

$$\psi^k(w_i) = \sum_j \langle \psi^k, s_{i,j} \rangle w_j.$$

Thus

$$\psi^k(z_i) = k^{d_i} u^{d_i} \sum_j \langle \psi^k, s_{i,j} \rangle u^{-d_j} z_j = \sum_j \alpha_{i,j,k} z_j.$$

Thus $A_k - k^m I$ is the matrix whose i th row gives $(\psi^k - k^m)(z_i)$ expressed in terms of the basis $\{z_1, \dots, z_r\}$. Then $\tilde{v}^{2m}(X)$ is the abelian group presented by a matrix obtained by stacking the matrices $A_k - k^m I$ for all k . Thus we see that the same matrix A , defined to be all matrices $A_k - k^m I$ stacked, yields $E_2^{0,2m+1}(X; \mathbf{Q}/\mathbf{Z})$ if we take the kernel of the linear transformation of free \mathbf{Q}/\mathbf{Z} -modules whose matrix is A , and it yields $\tilde{v}^{2m}(X)$ if we think of it as the presentation of an abelian group. Theorem 1.1 follows now from the following proposition.

Proposition 2.5. *If A is an integer matrix, let $K(A)$ denote the kernel of the linear transformation of free \mathbf{Q}/\mathbf{Z} -modules whose matrix is A , and let $G(A)$ denote the abelian group presented by A . Then there is an isomorphism of abelian groups $K(A) \approx (G(A))^\#$. The same is true if we are localized at a prime p .*

Proof. The matrix A can be brought to diagonal form (with rows and columns of 0's adjoined) by integer row and column operations. These operations consist of interchanging, multiplying by -1 , and adding a multiple of one to another. For the first interpretation of the matrix, they correspond to change-of-basis in the domain and range, and hence do not affect the kernel (up to isomorphism), while in the second interpretation they correspond to invertible change of generating set and to simplification of relations, and hence do not affect the isomorphism class of the group presented. Let D be a diagonal matrix with entries $d_i > 1$, m 1's, and n 0's on the diagonal, and possibly many additional rows of 0's. Then $K(D) \approx \bigoplus \mathbf{Z}/d_i \oplus (\mathbf{Q}/\mathbf{Z})^n$, while $G(D) \approx \bigoplus \mathbf{Z}/d_i \oplus (\mathbf{Z})^n$. Since $(\mathbf{Z})^\# \approx \mathbf{Q}/\mathbf{Z}$ and $(\mathbf{Z}/d)^\# \approx \mathbf{Z}/d$, the proposition follows. ■

Proof of Proposition 2.2. We must show that if $d(x) = \sum s_j \otimes y_j$, then $\bar{d}(x) \in U(X)$ if and only if $k^{d_j} \langle \psi^k, s_j \rangle$ is 2-integral for all j .

The following analysis of the unstable condition (i.e. the condition for being in $U(X)$) is based on [7, §7], which derived from [3]. Let \mathbf{K}_{2n} denote the $2n$ th space in the Ω -spectrum for K . There is a homotopy equivalence $\mathbf{K}_{2n} \rightarrow BU$, and the structure maps $\Sigma^2 \mathbf{K}_{2n} \rightarrow \mathbf{K}_{2n+2}$ are the Bott maps B . These combine to a tower

$$K_{2e}(BU) \xrightarrow{B_*} K_{2e+2}(BU) \xrightarrow{B_*} \cdots \xrightarrow{B_*} K_{2e+2d_j}(BU) \xrightarrow{B_*} \cdots \rightarrow K_{2e}(K),$$

with $K_{2e}(K)$ the direct limit of the tower. There is a similar tower after tensoring with \mathbf{Q} . Suppose $s_j \in K_{2e}(K) \otimes \mathbf{Q}$. Then $s_j \otimes y_j \in U(X)$ if and only if s_j pulls back to an element of $K_{2e+2d_j}(BU)$. This follows from the description of unstable comodules in the fourth paragraph after [7, 4.3].

For purposes of calculation, it is convenient to write the above tower in the equivalent form

$$K_0(BU) \xrightarrow{B_*} K_0(BU) \xrightarrow{B_*} \cdots \xrightarrow{B_*} K_0(BU) \xrightarrow{B_*} \cdots \rightarrow K_0(K).$$

In [3, 1.2-1.4], it is observed that $K_0(BU) \subset \mathbf{Q}[w]$ is a polynomial algebra generated by $\binom{w}{n} = w(w-1) \cdots (w-n+1)/n! \in \mathbf{Q}[w]$ for $n \geq 1$. Let A denote the free abelian group on the rational polynomials $\binom{w}{n}$. Then A consists of all rational polynomials $f(w)$ such that $f(n)$ is an integer for every integer n , so-called *numerical polynomials*. Then, still following [3], $B_*(f) - wf$ is decomposable for all $f \in A$, and B_* annihilates decomposables, from which it is deduced that $K_0(K) \approx w^{-1}A$, with an element f in the d th factor corresponding to $w^{-d}f$ in the limit.

It is also noted in [3, p.390] that $w = u^{-1}v \in K_0(K)$. This implies that $\langle \psi^k, w^n \rangle = k^n$. Thus $\langle \psi^k, f(w) \rangle = f(k)$, and so A consists of those polynomials $f(w)$ for which $\langle \psi^k, f(w) \rangle$ is integral for all k .

Now $s_j \in K_0(K) \otimes \mathbf{Q}$ is represented by a finite Laurent series $s_j(w)$ and pulls back to an element in the d_j th copy of $K_0(BU)$ if and only if $w^{d_j} s_j(w)$ is a numerical polynomial if and only if $\langle \psi^k, w^{d_j} s_j(w) \rangle$ is integral for all k . Since $\langle \psi^k, w^{d_j} s_j \rangle = k^{d_j} \langle \psi^k, s_j \rangle$, our desired conclusion follows. ■

3. PROOF OF THEOREM 1.5

We begin with a result culled from [16].

Proposition 3.1. *There are morphisms of Adams modules*

$$PK^1(SU(2n+1)) \xrightarrow{j^*} PK^1(Sp(n)) \xrightarrow{\theta} PK^1(Spin(2n+1))$$

$$\langle B_1, \dots, B_{2n} \rangle \longrightarrow \langle B'_1, \dots, B'_n \rangle \longrightarrow \langle B''_1, \dots, B''_{n-1}, D \rangle,$$

where $\langle -, \dots, - \rangle$ denotes the free abelian group on the indicated classes, $\theta(B'_i) = B''_i$ if $i < n$, $\theta(B'_1 + \dots + B'_n) = 2^{n+1}D$, and

$$j^*(B_i) = \begin{cases} B'_i & i \leq n \\ B'_{2n+1-i} & i > n. \end{cases}$$

Proof. We use Hodgkin's result ([14]) that $PK^1(G) \approx \langle \beta(\rho_1), \dots, \beta(\rho_l) \rangle$, where ρ_1, \dots, ρ_l are the fundamental representations of G , and $\beta(\rho)$ is the virtual vector bundle over ΣG associated to $\rho - \dim(\rho)$. Naylor ([16, p.151]) notes that the maps

$$Sp(n) \xrightarrow{k} SU(2n) \xrightarrow{i} SU(2n+1)$$

induce morphisms of representation rings

$$R(SU(2n+1)) \xrightarrow{i^*} R(SU(2n)) \xrightarrow{k^*} R(Sp(n))$$

satisfying

$$k^*(\mu'_i) = \begin{cases} \eta_i & i \leq n \\ \eta_{2n-i} & i > n \end{cases}$$

and $i^*(\mu_i) = \mu'_i + \mu'_{i-1}$, where η_i , $1 \leq i \leq n$, μ'_i , $1 \leq i \leq 2n-1$, and μ_i , $1 \leq i \leq 2n$, are the representations given by exterior power operations on a canonical representation. We have $\mu'_0 = \mu'_{2n} = 1$. Letting $B'_i = \beta(\eta_i) + \beta(\eta_{i-1})$ and $B_i = \beta(\mu_i)$ yields the desired result about j^* , which is the composite $k^* \circ i^*$.

Naylor also considers the composite

$$j' : Spin(2n+1) \rightarrow SO(2n+1) \rightarrow SU(2n+1).$$

We let λ_i denote the exterior powers of the canonical unitary representation of $Spin(2n+1)$, Δ the Spin representation, $B''_i = \beta(\lambda_i)$, and $D = \beta(\Delta)$. Naylor shows that

$$j'^*(B_i) = \begin{cases} B''_i & i \leq n \\ B''_{2n+1-i} & i > n \end{cases}$$

and $2^{n+1}D = B_1'' + \cdots + B_n''$. Since $\ker(j^*) = \ker(j'^*)$, there is a homomorphism of abelian groups

$$PK^1(Sp(n)) \xrightarrow{\theta} PK^1(Spin(2n+1))$$

defined by $\theta(B_i') = B_i''$ for $i \leq n$, and it will be an Adams-module homomorphism since j^* and j'^* are Adams-module homomorphisms with j^* surjective. ■

Next we relate the above basis of $PK^1(SU(n))$ with one used by Bousfield ([9]).

Proposition 3.2. *There is an isomorphism of Adams modules*

$$PK^1(SU(n)) \approx \widetilde{K}^0(CP^{n-1}). \quad (3.3)$$

Let ξ_k denote the element of $PK^1(SU(n))$ which corresponds to $\xi^k - 1$ under this isomorphism. Here ξ denotes the Hopf bundle over CP^{n-1} . Then

$$B_j = \sum (-1)^{k+1} \binom{n}{j-k} \xi_k.$$

Proof. If I denotes the augmentation ideal of $R(G)$, then there is an isomorphism $I/I^2 \approx PK^1(G)$ under which $(-1)^{k+1}\lambda^k$ corresponds to ψ^k . (See, e.g., [11, 2.1ff].) We first claim that under this isomorphism and (3.3) (which is well-known) $\{\theta_n - n\} \in I/I^2$ corresponds to $\{\xi - 1\} \in \widetilde{K}^0(CP^{n-1})$. To see this, we use a result in [15, p.206] which states that there is a commutative diagram

$$\begin{array}{ccc} CP^{n-1} & \xrightarrow{f} & BU \\ j \downarrow & & \downarrow B \\ \Omega SU(n) & \xrightarrow{\Omega i} & \Omega SU \end{array}$$

in which j is a canonical map, i is the inclusion, B the Bott map, and f satisfies $f^*(c_i) = 0$ unless $i = 1$, and $f^*(c_1)$ is the usual generator of $H^2(CP^{n-1})$. This implies that f classifies ξ . The isomorphism (3.3) is defined using the maps j and B . Hence ξ corresponds under this isomorphism to the map i , which is the canonical representation θ_n .

Since $\psi^k(\xi) = \xi^k$, we obtain that

$$\xi_k = (-1)^{k+1} \lambda^k(\theta_n - n) = (-1)^{k+1} \sum_{j=0}^k \binom{-n}{k-j} \lambda^j(\theta_n).$$

But $\lambda^j(\theta_n) = B_j$. To invert this equation, we take a formal sum of the equations for all values of k , using powers of an indeterminate x . We have

$$\sum (-1)^{k+1} \xi_k x^k = \sum x^k \sum_{j=0}^k \binom{-n}{k-j} B_j = (1+x)^{-n} \sum x^j B_j$$

and hence

$$\sum x^j B_j = \sum_i \binom{n}{i} x^i \sum_k (-1)^{k+1} \xi_k x^k = \sum x^j \sum_k \binom{n}{j-k} (-1)^{k+1} \xi_k,$$

as desired. \blacksquare

The following corollary is immediate from the description of j^* in Proposition 3.1 along with Proposition 3.2.

Corollary 3.4. $PK^1(Sp(n)) \approx \langle \xi_1, \xi_2, \dots : R_{n+1}, \dots, R_{2n}, S_j, j > 2n \rangle$, where

$$R_j = \sum (-1)^{k+1} \binom{2n+1}{j-k} \xi_k - \sum (-1)^{k+1} \binom{2n+1}{2n+1-j-k} \xi_k,$$

$$S_j = \sum (-1)^k \binom{j}{k} \xi_k, \text{ and } \psi^t \xi_k = \xi_{kt}.$$

Note that the relations allow one to express each ξ_i with $i > n$ in terms of ξ_1, \dots, ξ_n . From Corollary 3.4, we easily deduce the following result.

Corollary 3.5. $v^{2m}(Sp(n)) \approx \mathbf{Z}/2^{eSp'(m,n)}$, where

$$eSp'(m,n) := \min(\nu(R_{m,n+1}), \dots, \nu(R_{m,2n}), \nu(S'_{m,j}), j > 2n),$$

with $R_{m,j}$ and $S'_{m,j}$ defined by

$$R_{m,j} = \sum_{k \text{ odd}} \binom{2n+1}{j-k} k^m - \sum_{k \text{ odd}} \binom{2n+1}{2n+1-j-k} k^m,$$

$$S'_{m,j} = \sum_{k \text{ odd}} \binom{j}{k} k^m.$$

Proof. The sums are taken over all odd values of k for which the binomial coefficients are nonzero. The generator of the group is ξ_1 , and the relations are obtained from those in 3.4 by $\xi_k = \psi^2(\xi_{k/2}) \equiv 0$ if k is even, while if k is odd, $\xi_k = \psi^k(\xi_1) \equiv k^m \xi_1$. Here we are using Corollary 3.9 to imply that $\psi^k \equiv k^m$ in $v^{2m}(-)$ for all odd k . \blacksquare

Because of Theorems 1.1 and 1.4 and Corollary 3.5, we have

Proposition 3.6. $eSp'(m, n) = eSp(m, n)$; i.e.,

$$\min(\nu(R_{m,n+1}), \dots, \nu(R_{m,2n}), \nu(S_{m,j}), j > 2n) = \min(\nu(S_{m,j}), j > 2n),$$

and so the relations $R_{m,j}$ are superfluous in $eSp'(m, n)$

We should remark on the minor difference between $S_{m,j}$ in 1.4 and $S'_{m,j}$ in 3.5, in that one involves a sum over all values of k , while the other involves a sum only over odd values of k . As remarked in Section 1, we only consider values of m large enough that summands which are multiples of 2^m would not affect divisibility by 2 of the sum. Thus $\nu(S_{m,j}) = \nu(S'_{m,j})$ for such values of m .

Next we explain why the relations $\psi^3 - 3^m$ and $\psi^{-1} - (-1)^m$ imply relations $\psi^k - k^m$ for all odd integers k . We contrast with the situation when we are localized at an odd prime. The difference is due to the following result, which is stated in [1, 2.9].

Proposition 3.7. *If p is an odd prime, then the multiplicative group of units $(\mathbf{Z}/p^n)^\times$ is cyclic of order $(p-1)p^{n-1}$. If r generates $(\mathbf{Z}/p^2)^\times$, then any integer congruent to $r \bmod p^2$ generates $(\mathbf{Z}/p^n)^\times$. The group $(\mathbf{Z}/2^n)^\times$ is the direct product of the subgroup consisting of ± 1 and the subgroup of classes congruent to 1 mod 4; the latter subgroup is cyclic of order 2^{n-2} generated by any integer congruent to 5 mod 8.*

This implies that, for any n , any odd integer is congruent mod p^n to $\pm 3^e$ for some e . We also need the following result of Adams.

Proposition 3.8. ([2, 5.1]) *If X is a finite complex, there exists an integer e so that for all k and m , $\psi^k \equiv \psi^{k+m^e}$ in $K(X)/mK(X)$.*

We will use the following consequence of these results.

Corollary 3.9. *Let X be a finite complex, and if p is odd let r generate $(\mathbf{Z}/p^2)^\times$. Let*

$$Q_m = \begin{cases} PK^1(X; \mathbf{Z}_{(p)})/(\text{im}(\psi^r - r^m)) & \text{if } p \text{ is odd} \\ PK^1(X; \mathbf{Z}_{(2)})/(\text{im}(\psi^3 - 3^m, \psi^{-1} - (-1)^m)) & \text{if } p = 2. \end{cases}$$

Then $\psi^s = s^m$ in Q_m for all $s \not\equiv 0 \bmod p$.

Proof. We consider just the case $p = 2$. The case when p is odd is similar and slightly easier. We show $\psi^s \equiv s^m \bmod 2^n$ for all positive integers n . This implies that they are equal in the finitely generated $\mathbf{Z}_{(2)}$ -module.

We first show that Proposition 3.8 is valid for $K^1(-)$ provided k and m are relatively prime. Adams' result dealt with $K^0(-)$. The validity in $K^{-1}(-)$ is immediate since $K^{-1}(X) = K^0(\Sigma X)$. Since ψ^k in $K^{-1}(-)$ corresponds to $k\psi^k$ in $K^1(-)$, the periodicity in $K^1(-)$ follows, as long as k is a unit mod m .

Choose an integer e that works for X in Proposition 3.8. Use Proposition 3.7 to find an integer ℓ and $\epsilon = 0$ or 1 so that $s \equiv (-1)^\epsilon 3^\ell \pmod{2^{ne}}$. Then, for some k , we have, in $Q_m \pmod{2^n}$,

$$\psi^s = \psi^{(-1)^\epsilon 3^\ell + k2^{ne}} \equiv \psi^{(-1)^\epsilon} \psi^{3^\ell} \equiv ((-1)^\epsilon)^m (3^\ell)^m \equiv s^m.$$

■

The following result is immediate from Propositions 3.1 and 3.2.

Proposition 3.10. *The group $PK^1(\text{Spin}(2n+1))$ has a subgroup isomorphic to $PK^1(\text{Sp}(n))$, as described in Corollary 3.4, and an additional generator D satisfying*

$$2^{n+1}D = \sum_{j=1}^n \sum_{k=1}^j (-1)^{k+1} \binom{2n+1}{j-k} \xi_k. \quad (3.11)$$

The Adams operations are determined by $\psi^t \xi_k = \xi_{kt}$.

The next two theorems, 3.15 and 3.18, determine $v^{2m}(\text{Spin}(2n+1))$. Two of the relations, \mathcal{R}_2 and \mathcal{R}_3 , are defined in terms of the following algorithms. The algorithms will be justified in the proof of 3.15 and computed in 3.18. We use the relations R_j , $n+1 \leq j \leq 2n$, and S_j , $2n < j \leq 3n$, of Corollary 3.4. Note that each equals $\pm \xi_j$ mod terms with smaller subscripts of ξ .

For \mathcal{R}_2 , begin with

$$\sum_{j=1}^n \sum_{k=1}^j (-1)^{k+1} \binom{2n+1}{j-k} \xi_{2k}, \quad (3.12)$$

and subtract multiples of R_{2n}, \dots, R_{n+1} to eliminate $\xi_{2n}, \dots, \xi_{n+1}$. This results in a linear expression E in ξ_1, \dots, ξ_n which, we will show in (4.2), equals $(-1)^{n+1} 2^n \xi_n$ plus lower terms. Write (3.11) in the form

$$2^{n+1}D + (-1)^n \xi_n + \sum_{k=1}^{n-1} (-1)^k \xi_k \sum_{i=0}^{n-k} \binom{2n+1}{i}. \quad (3.13)$$

Add 2^n times (3.13) to E to eliminate ξ_n . The resulting linear expression in ξ_1, \dots, ξ_{n-1} , and D is divisible by 2^{n+1} . Divide it by 2^{n+1} , and then replace ξ_k by k^m (resp. 0) if k is odd (resp. even) to get \mathcal{R}_2 .

For \mathcal{R}_3 , begin with

$$\sum_{j=1}^n \sum_{k=1}^j (-1)^{k+1} \binom{2n+1}{j-k} \xi_{3k}, \quad (3.14)$$

and subtract multiples of $S_{3n}, \dots, S_{2n+1}, R_{2n}, \dots, R_{n+1}$, and of (3.13) to eliminate $\xi_{3n}, \dots, \xi_{n+1}, \xi_n$. This results in a linear expression in ξ_1, \dots, ξ_{n-1} , and D which is divisible by 2^{n+1} . Divide it by 2^{n+1} and replace ξ_k by k^m or 0 as before to get an expression E' . The relation \mathcal{R}_3 is $3^m D - E'$.

Theorem 3.15. *The abelian group $v^{2m}(\text{Spin}(2n+1))$ has generators ξ_1 and D with relations*

$$2^{n+1}D = \left(\sum_{j=1}^n \sum_{\text{odd } k} \binom{2n+1}{j-k} k^m \right) \xi_1, \quad (3.16)$$

$2^{eSp(m,n)} \xi_1, \mathcal{R}_2$ and \mathcal{R}_3 .

Proof. The second relation is the one obtained in Propositions 3.5 and 3.6. This relation due to Adams operations on ξ_i -classes is still present in $v^{2m}(\text{Spin}(2n+1))$. The first relation is the definition of D , together with $\xi_k \sim \psi^k \xi_1 \sim k^m \xi_1$ if k is odd, and $\xi_k \sim \psi^k \xi_1 \sim 0$ if k is even.

The relations \mathcal{R}_2 and \mathcal{R}_3 are consequences of $\psi^2 D \sim 0$ and $\psi^3 D \sim 3^m D$, respectively. We evaluate $\psi^2(2^{n+1}D)$ and $\psi^3(2^{n+1}D)$ using (3.11) and $\psi^\ell \xi_k \sim \xi_{k\ell}$. The relations are reduced using R_j and S_j . Since $\psi^k D$ must be an integral combination of the generators ξ_1, \dots, ξ_{n-1} , and D , it follows that $\psi^2(2^{n+1}D)$ and $\psi^3(2^{n+1}D)$, when expressed in terms of these classes, must be divisible by 2^{n+1} . Performing the division yields the desired relations for $\psi^2(D)$ and $\psi^3(D)$.

By Corollary 3.9, we need only to append the relation $\psi^{-1}(D) \sim (-1)^m D$ in order to achieve all relations $\psi^k(D) \sim k^m D$ for all odd k . However, by Proposition 3.17, this relation adds no information if m is odd, while if m is even, which will be treated at the end of this section, it implies $2D \sim 0$. ■

Proposition 3.17. $\psi^{-1} = -1$ in $PK^1(\text{Spin}(2n+1))$.

Proof. First note that $\psi^{-1} = (-1)^m$ in $PK^1(S^{2m+1})$. We use this to prove, by induction on n , that $\psi^{-1} = -1$ in $PK^1(Sp(n))$.

Indeed, consider the short exact sequence

$$0 \rightarrow PK^1(S^{4n-1}) \xrightarrow{p^*} PK^1(Sp(n)) \xrightarrow{i^*} PK^1(Sp(n-1)) \rightarrow 0,$$

and assume $\psi^{-1} = -1$ in $PK^1(Sp(n-1))$. Let y generate $\text{im}(p^*)$. Note $\psi^{-1}(y) = -y$. Let $x \in PK^1(Sp(n))$. By naturality of ψ^{-1} and the induction hypothesis, we have $i^*(\psi^{-1}(x) + x) = 0$, and hence $\psi^{-1}(x) + x = \alpha y$ for some integer α . Apply ψ^{-1} and use $\psi^{-1}\psi^{-1} = 1$ to obtain $\alpha y = -\alpha y$, and hence $\alpha = 0$.

Now we use the morphism $PK^1(Sp(n)) \xrightarrow{\theta} PK^1(Spin(2n+1))$ considered in Proposition 3.1 to deduce the proposition. Since $2^{n+1}D \in \text{im}(\theta)$, we obtain $\psi^{-1}(2^{n+1}D) = -2^{n+1}D$, and hence $\psi^{-1}(D) = -D$ since there is no torsion in $PK^1(Spin(2n+1))$. All other elements are in the image from $PK^1(Sp(n))$. ■

The algorithm distills down to the following closed form for the relations in Theorem 3.15, which was observed from extensive computer calculations by the second author and then proved (see Section 4) utilizing a number of elaborate combinatorial arguments.

Theorem 3.18. *$v^{2m}(Spin(2n+1))$ is the abelian group with generators ξ_1 and D and four relations as below:*

$$2^{eSp(m,n)}\xi_1,$$

$$\left(\sum_{\text{odd } k} k^m \sum_{i=0}^{n-k} \binom{2n+1}{i} \right) \xi_1 - 2^{n+1}D, \quad (3.19)$$

$$\left(\sum_{\text{odd } k} k^m \sum_{t \geq 0} \binom{2n+2}{n-1-k-4t} \right) \xi_1 - 2^n D, \quad (3.20)$$

and

$$\left(2^n \sum_{\text{odd } k} k^m \sum_{t \geq 0} \binom{2n+1}{n-1-k-3t} \right) \xi_1 - \frac{1}{3}(2^{2n+1} + 1 - 3^{m+1})D. \quad (3.21)$$

If m is odd and $n \neq 4$, the coefficients of ξ_1 in (3.19) and (3.20) are divisible by 2^n .

From this, we easily deduce Theorem 1.5. Subtract 2 times (3.20) from (3.19) to obtain the relation $R_1(m, n)$ of 1.5 on the generator ξ_1 . Note that the exponent of 2 in the coefficient of D in (3.21) is $\min(2n+1, \nu(m+1)+2)$ if m is odd. If $n \leq \nu(m+1)+2$, divide (3.20) by 2^n to split off $\mathbf{Z}/2^n$, the second summand in 1.5, and then add an appropriate multiple of (3.20) to (3.21) to eliminate D , yielding the relation $R_2(m, n)$ of 1.5 on ξ_1 . The last part of Theorem 3.18 is necessary in order to know that we can split off $\mathbf{Z}/2^n$.

Similarly, if $\nu(m+1)+2 < n$, use (3.21) to split off $\mathbf{Z}/2^{\nu(m+1)+2}$, and then subtract an appropriate multiple of (3.21) from (3.20) to obtain $R_2(m, n)$. This completes the proof of Theorem 1.5 when m is odd.

To prove that $v^{2m}(\text{Spin}(2n+1)) \approx \mathbf{Z}/2 \oplus \mathbf{Z}/2$ when m is even, we begin by showing that $eSp'(m, n) = 1$ in Corollary 3.5 when m is even. It suffices to show that $R_{m, 2n} \equiv 2 \pmod{4}$ and that $R_{m, j}$ and $S'_{m, j}$ are even for all relevant j . Since $k^m \equiv 1 \pmod{4}$ if k is odd and m even, we have, mod 4,

$$S'_{m, j} \equiv \sum_{k \text{ odd}} \binom{j}{k} = 2^{j-1},$$

$$R_{m, 2n} \equiv \sum_{k \text{ odd}} \binom{2n+1}{k} - 2 \binom{2n+1}{2n+1} = 2^{2n} - 2,$$

and other $R_{m, j}$ are congruent to 2^{2n} minus 2 times a sum of binomial coefficients.

As observed in the proof of 3.15, ψ^{-1} implies $2D \sim 0$ when m is even. Thus it remains to show that the coefficients of relations in Theorem 3.18 are even when m is even, which reduces to showing

$$\sum_{k \text{ odd}} \sum_{i=0}^{n-k} \binom{2n+1}{i} \equiv 0 \pmod{2} \quad (3.22)$$

and

$$\sum_{k \text{ odd}} \sum_{t \geq 0} \binom{2n+2}{n-k-1-4t} \equiv 0 \pmod{2}. \quad (3.23)$$

The sum in (3.22) is $\sum_{i=0}^{n-1} \binom{2n+1}{i} \left\lceil \frac{n-i+1}{2} \right\rceil$. If n is even, terms $2j$ and $2j+1$ in this sum will have the same parity. If $n \equiv 3 \pmod{4}$, terms $2j+1$ and $2j+2$ will have the same parity. If $n \equiv 1 \pmod{4}$, the only odd terms will occur in pairs $i = 8j$ and $i = 8j+3$.

The sum in (3.23) is $\sum \binom{2n+2}{n-2j} \left\lfloor \frac{j+1}{2} \right\rfloor$. If n is odd, all coefficients $\binom{2n+2}{n-2j}$ are even, while if $n = 2N$, this becomes $\sum \binom{2N+1}{N-j} \left\lfloor \frac{j+1}{2} \right\rfloor \pmod 2$, which equals the first sum, and hence is even. This completes the proof of Theorem 1.5.

4. PROOF OF THEOREM 3.18

In this section, we perform the algorithm in Theorem 3.15 and obtain Theorem 3.18. First note that (3.19) is an elementary manipulation of (3.16).

We begin by deriving (3.20). Let M be a matrix whose rows present the relations R_{2n}, \dots, R_{n+1} of 3.4 expressed in terms of ξ_{2n}, \dots, ξ_1 . Then

$$M_{i,j} = (-1)^{j-i} \binom{2n+1}{j-i} + (-1)^{i+j} \binom{2n+1}{j+i-2n-1}.$$

Let M_L (resp. M_R) denote the left (resp. right) half of M .

Let $\sigma_j := \sum_{k=0}^j \binom{2n+1}{k}$. Let P denote a row vector of length $2n$ whose $(2j+1)$ st entry is $(-1)^{n+1+j} \sigma_j$ for $0 \leq j < n$, with other entries 0. This represents (3.12). Write $P = (P_L | P_R)$, with P_L and P_R of length n . We wish to add multiples of the rows of M to P to annihilate P_L . This is facilitated by first replacing M by the equivalent set of relations $M_L^{-1}M = [I | M_L^{-1}M_R]$. The linear expression E in the algorithm for \mathcal{R}_2 described before Theorem 3.15 corresponds to $P_R - P_L M_L^{-1}M_R$. We will show that this equals

$$(-1)^{n+1} 2^n \sum_{j=0}^{n-1} (-1)^j \binom{2n+1}{j} V_j, \quad (4.1)$$

where V_j is a vector of length n which begins with j 0's followed by

$$1, 1, -1, -1, 1, 1, -1, \dots$$

The expression corresponding to (4.1) is

$$2^n \left(\sum_{k=1}^n (-1)^{k+1} \xi_k \left(\sum_{j=0}^{n-k} \binom{2n+1}{j} - 2 \sum_{t \geq 0} \binom{2n+2}{n-1-k-4t} \right) \right). \quad (4.2)$$

We have used

$$\binom{2n+2}{a} = \binom{2n+1}{a} + \binom{2n+1}{a-1} \quad (4.3)$$

to show that this corresponds to (4.1). Adding 2^n times (3.13) to (4.2) and replacing ξ_k by k^m or 0 yields -2^{n+1} times (3.20), as desired.

To complete the proof of (3.20), it remains to prove (4.1). Let $c_i = \binom{2n+1}{i}$ and $d_i = \binom{2n+i}{i}$. Consideration of generating functions implies that for any positive integer m

$$\sum_{j=0}^m (-1)^j d_j c_{m-j} = 0 \quad (4.4)$$

and that the (i, j) th entry of M_L^{-1} is d_{j-i} . Using these facts, one can easily prove the following lemma.

Lemma 4.5. $M_L^{-1} M_R = \sum_{j=1}^n D_j C_j$, where D_j is a column vector of length n with t^{th} entry $d_{n+1-t-j} - d_{n+j-t}$, and C_j is a row vector of length n with t^{th} entry $(-1)^{t-j} c_{t-j}$.

To simplify notation, we shall complete the proof of (4.1) when n is odd. The proof when n is even is virtually the same. Let $n = 2e + 1$. We shall prove the following lemma.

Lemma 4.6. For $1 \leq j \leq n$,

$$-P_L D_j = (-1)^{[(j-1)/2]} 2^n + \sum_{u=1}^{[j/2]} (-1)^{e+u+1} \sigma_{e+u} d_{j-2u}.$$

Then the entry in the t^{th} position of $-P_L M_L^{-1} M_R$ is

$$(-P_L \sum_j D_j C_j)_t = A_{1,t} + A_{2,t},$$

where

$$A_{1,t} = 2^n \sum_{j=1}^t (-1)^{[(j-1)/2]} (-1)^{t-j} c_{t-j},$$

which equals the t^{th} entry of (4.1), and

$$A_{2,t} = \sum_{j=1}^t (-1)^{t-j} c_{t-j} \sum_{u=1}^{[j/2]} (-1)^{e+u+1} \sigma_{e+u} d_{j-2u}.$$

Using (4.4), one can check that

$$A_{2,t} = \begin{cases} 0 & t \text{ odd} \\ (-1)^{e+T+1} \sigma_{e+T} & t = 2T. \end{cases}$$

Thus $P_R + A_2 = 0$, and hence $P_R - P_L M_L^{-1} M_R$ equals (4.1), as claimed. Once we have given the proof of Lemma 4.6, we will be done with the proof of the \mathcal{R}_2 -part of Theorem 3.18.

Proof of Lemma 4.6. We use generating functions. The coefficients of x^0, \dots, x^{n-1} in $p(x) := (1 - x^2)^{2n+1}/(1 + x^2)$ give the entries of P_L , while $q(x) := (1 - x)^{-(2n+1)} = \sum d_i x^i$. Note that $p(x)q(x) = (1 + x)^{2n+1}/(1 + x^2)$. We obtain

$$\begin{aligned} -P_L D_j &= \sum_{t=1}^n \text{coef}(x^{t-1}, p(x)) \cdot (\text{coef}(x^{n+j-t}, q(x)) - \text{coef}(x^{n-j+1-t}, q(x))) \\ &= \text{coef}(x^{n+j-1}, p(x)q(x)) - \text{coef}(x^{n-j}, p(x)q(x)) \\ &\quad - \sum_{t>n} \text{coef}(x^{t-1}, p(x)) \cdot d_{n+j-t}. \end{aligned} \tag{4.7}$$

The sum subtracted at the end is due to the truncation of D_j ; one easily checks that this equals the u -sum in Lemma 4.6. Finally we observe that, if we employ the symmetry property of binomial coefficients, then the expression in line (4.7) equals the following expression if $j \equiv 1$ or $2 \pmod{4}$, and equals the negative of this expression otherwise.

$$\binom{2n+1}{n} - \binom{2n+1}{n-1} - \binom{2n+1}{n-2} + \binom{2n+1}{n-3} + \binom{2n+1}{n-4} - \binom{2n+1}{n-5} - \dots$$

This expression equals 2^n , as can be seen by expanding $(1 - i)^{2n+1}$ or by induction on n . ■

The derivation of (3.21) is similar to that of (3.20) just performed, although somewhat more complicated. Replacing the matrix M used in (3.20) is a matrix N whose rows present the relations $S_{3n}, \dots, S_{2n+1}, R_{2n}, \dots, R_{n+1}$ of 3.4 in terms of ξ_{3n}, \dots, ξ_1 . Then

$$N_{i,j} = \begin{cases} (-1)^{j-i} \binom{3n+1-i}{j-i} & 1 \leq i \leq n \\ (-1)^{j-i} \binom{2n+1}{j-i} + (-1)^{i+j} \binom{2n+1}{j+i-4n-1} & n < i \leq 2n. \end{cases}$$

Divide N into $n \times n$ submatrices, which we name as follows:

$$N = \begin{pmatrix} T_1 & U_1 & U_2 \\ 0 & T_2 & U_3 \end{pmatrix}.$$

We perform row operations on N to reduce it to

$$N' = \begin{pmatrix} I & 0 & B_2 - B_1 W \\ 0 & I & W \end{pmatrix},$$

where $B_i = T_1^{-1}U_i$, and $W = T_2^{-1}U_3 = \sum_{j=1}^n D_j C_j$, exactly as in Lemma 4.5.

Let σ_j, c_i, d_i, C_j , and D_j be as in the preceding proof, with $\sigma_j = 0$ if j is not an integer. Let Q denote a row vector of length $3n$ whose $(3j+1)$ st entry is $(-1)^{n+1+j}\sigma_j$

for $0 \leq j < n$, with other entries 0. This represents (3.14). Write $Q = (Q_0|Q_1|Q_2)$ with each Q_i of length n . We use N' to reduce Q to $(0|0|Q'_2)$, where

$$Q'_2 = Q_2 - Q_0(B_2 - B_1W) - Q_1W. \quad (4.8)$$

We will show this equals

$$\begin{aligned} & (-1)^n \left(\frac{1}{3}(2^{2n+1} + 1)(-\sigma_0, \sigma_1, -\sigma_2, \sigma_3, \dots, \sigma_{n-1}) \right. \\ & \quad \left. + 2^{2n+1}(0, -c_0, c_1, -c_2, c_3 + c_0, -(c_4 + c_1), \dots) \right). \end{aligned} \quad (4.9)$$

Thus the vector (4.9) expresses $\psi^3(2^{n+1}D)$ in terms of ξ_n, \dots, ξ_1 as

$$\frac{1}{3}(2^{2n+1} + 1) \sum_{k=1}^n (-1)^{k+1} \xi_k \sum_{i=0}^{n-k} \binom{2n+1}{i} + 2^{2n+1} \sum_{k=1}^{n-1} (-1)^k \xi_k \sum_{t \geq 0} \binom{2n+1}{n-1-k-3t}. \quad (4.10)$$

We add $\frac{1}{3}(2^{2n+1} + 1)$ times (3.13) to eliminate ξ_n and get

$$\frac{1}{3}(2^{2n+1} + 1)2^{n+1}D + 2^{2n+1} \sum_{k=1}^{n-1} (-1)^k \xi_k \sum_{t \geq 0} \binom{2n+1}{n-1-k-3t}.$$

Divide by 2^{n+1} , replace ξ_k by k^m or 0, and subtract from $3^m D$; this yields exactly (3.21).

It remains to show that Q'_2 equals (4.9). We show this when $n \equiv 0 \pmod{6}$; other congruences can be handled similarly. Let $r = n/3$. It is not difficult to prove that

$$(B_1)_{i,j} = (-1)^j \binom{n+j-i-1}{n-i} \binom{3n+1-i}{n+j-i} \text{ and } (B_2)_{i,j} = (-1)^j \binom{2n+j-i-1}{n-i} \binom{3n+1-i}{2n+j-i}.$$

We obtain that, for $0 \leq t < n$, the $(t+1)$ st entry of Q'_2 is

$$(-1)^{t+1} \sigma_{2r+t/3} - \sum_{u=0}^{r-1} (-1)^{u+t} \sigma_u \binom{2n+t-3u-1}{n+t} \binom{3n-3u}{n-t} \quad (4.11)$$

$$+ \sum_{s=0}^{n-1} \sum_{u=0}^{r-1} (-1)^{u+s} \sigma_u \binom{n+s-3u-1}{s} \binom{3n-3u}{2n-s} \quad (4.12)$$

$$\cdot \sum_{j=0}^t (d_{n-s+j-t-1} - d_{n+t-j-s}) (-1)^j c_j$$

$$+ \sum_{u=0}^{r-1} (-1)^u \sigma_{r+u} \sum_{j=0}^t (d_{n-3u-1+j-t} - d_{n+t-j-3u}) (-1)^j c_j. \quad (4.13)$$

It follows from (4.4) that the j -sum in (4.12) is 0 for $s \geq n$, except that it is -1 when $s = n + t$. Hence (4.12) can be made into a sum over all $s \geq 0$ with the sum-part of (4.11) incorporated into it.

Next we note that if $u = r + \delta$ with $\delta \geq 0$ in (4.13), then, again using (4.4), the j -sum in (4.13) is nonzero only when $\delta = t/3$, in which case it is -1 . Thus (4.13) can be extended to $u \geq 0$ if the initial term of (4.11) is included. Similarly we note that if $u = r + \Delta$ with $\Delta \geq 0$ in (4.12), then $\binom{n+s-3u-1}{s} \binom{3n-3u}{2n-s}$ is nonzero only when $s = 3\Delta$, in which case it is $(-1)^\Delta$. Thus the modified (4.13) can be incorporated into (4.12) by extending the u -sum of (4.12) to $u \geq 0$. Now the $(t+1)$ st entry of Q'_2 has been simplified to

$$\sum_{s \geq 0} \sum_{u \geq 0} (-1)^{u+s} \sigma_u \binom{n+s-3u-1}{s} \binom{3n-3u}{2n-s} \sum_{j=0}^t (d_{n-s+j-t-1} - d_{n+t-j-s}) (-1)^j c_j. \quad (4.14)$$

We will prove the following lemma at the end of this section.

Lemma 4.15. *If $n \geq 0$, and v and w are arbitrary integers, then*

$$\sum_{s \geq 0} (-1)^s \binom{n+s-1-w}{s} \binom{3n-w}{2n-s} \binom{3n-s+v}{2n} = \binom{4n-w+v}{2n}.$$

Using this with $w = 3u$, (4.14) reduces to

$$\sum_{j=0}^t (-1)^j c_j \sum_{u \geq 0} (-1)^u \sigma_u \left(\binom{4n-3u+j-t-1}{2n} - \binom{4n-3u+t-j}{2n} \right). \quad (4.16)$$

Viewing $\binom{4n-3u+\epsilon}{2n}$ as $(-1)^{u+\epsilon} \text{coef}(x^{2n-3u+\epsilon}, (1+x)^{-(2n+1)})$, and σ_u as

$$\text{coef}(x^{3u}, (1+x^3)^{2n+1}/(1-x^3)),$$

we obtain a simplification of (4.16) to

$$(-1)^{t+1} \sum_{j=0}^t c_j \left(\text{coef}(x^{2n+j-t-1}, \frac{(1-x+x^2)^{2n+1}}{1-x^3}) + \text{coef}(x^{2n+t-j}, \frac{(1-x+x^2)^{2n+1}}{1-x^3}) \right). \quad (4.17)$$

We will prove the following lemma at the end of this section.

Lemma 4.18. *Let $f_m(x) = (1-x+x^2)^{m+1}/(1-x^3)$. Then for $m \geq 0$ and $j \geq 0$,*

$$\text{coef}(x^{m-j-1}, f_m(x)) + \text{coef}(x^{m+j}, f_m(x)) = \begin{cases} (1+2(-2)^m)/3 & j \equiv 0, 2 \pmod{3} \\ (1+2(-2)^{m+1})/3 & j \equiv 1 \pmod{3}. \end{cases}$$

We obtain that (4.17) equals

$$(-1)^{t+1} \left(\frac{1}{3}(2^{2n+1} + 1) \sum_{j=0}^t c_j - 2^{2n+1} \sum_{\substack{0 \leq j \leq t \\ t-j \equiv 1 \pmod{3}}} c_j \right),$$

which equals the component of (4.9) in position $t+1$, as desired, completing the proof of (3.21), as described surrounding (4.9).

Next we prove the portion of Theorem 3.18 which states that

$$\sum_{\text{odd } k} k^m \sum_{t \geq 0} \binom{2n+2}{n-1-k-4t} \quad (4.19)$$

is divisible by 2^n if m is odd and $n \neq 4$. This occurs in (3.20). We accomplish this by showing later in this section that, if $m = 2a + 1$, then (4.19) equals

$$\binom{n-1}{1} 2^{2n-4} + \sum_{j=2}^{n/2} \binom{n-j}{j} 2^{2n-4j} \sum_{i \geq j-1} 8^i \binom{a}{i} \sum_{t=0}^{j-2} (-1)^t \binom{2j-1}{t} (2j-2t-1) \binom{j-t}{2}^i. \quad (4.20)$$

Then we note that the first term of (4.20) is divisible by 2^n provided $n \geq 3$. For each value of $j \geq 2$, the j -summand is nonzero for $n \geq 2j$ and is clearly divisible by $2^{2n-4j+3(j-1)}$, which is $\geq 2^n$ provided $n \geq j+3$. The case $n = 4$ and $j = 2$ is the only time that this gives a nonzero term which might not be divisible by 2^n ; it is what causes the restriction $n \neq 4$ in the last part of Theorem 3.18.

The expression (4.20), while not as elegant as (4.19), is often more useful in performing specific calculations of $\nu(-)$, such as in Proposition 1.6. Before proving (4.20), we handle the case $n = 4$, which was excluded from Theorem 1.5.

Proposition 4.21.

$$v^{2m}(\text{Spin}(9)) \approx \begin{cases} \mathbf{Z}/2 \oplus \mathbf{Z}/2 & m \text{ even} \\ \mathbf{Z}/8 \oplus \mathbf{Z}/2^{e(m)} & m \text{ odd,} \end{cases}$$

where

$$e(m) = \begin{cases} \min(\nu(m-5) + 2, 6) & \text{if } m \equiv 1 \pmod{4} \\ \min(\nu(m-7) + 2, 8) & \text{if } m \equiv 3 \pmod{4}. \end{cases}$$

Proof. The numbers $eSp(m, n)$ are as in [6, 5.1] (suitably interpreted), and large enough as to not matter here. Let $P = 3^m - 3$. The other three relations of 3.18 are $(160 + 10P)\xi_1 - 32D$, $(48 + P)\xi_1 - 16D$, and $16(39 + P)\xi_1 + (P - 168)D$. If $m \equiv 3$

mod 4, then $P \equiv 8 \pmod{16}$, so we get a $\mathbf{Z}/8$ due to $(48 + P)\xi_1$. (This is the anomaly.) If $m \equiv 1 \pmod{4}$, $P \equiv 0 \pmod{16}$, and we get a $\mathbf{Z}/8$ due to $(P - 168)D$.

If $m \equiv 1 \pmod{4}$, let $Q = 3(3^{m-5} - 1)$, so that $\nu(Q) = \nu(m - 5) + 2$. Replace P by $3^4Q + 3^5 - 3$. Subtract multiples of the third relation from the other two so as to eliminate the D -term from each. The relations obtained are, up to odd multiples, $2^6 + 2Q + Q^2/4$ and $2^8 + Q + Q^2/8$. For $\nu(Q) \geq 4$, the minimum of the exponent of 2 of these two expressions is $\min(6, \nu(Q))$, as claimed. The case $m \equiv 3 \pmod{4}$ is handled similarly. ■

Now we establish the equivalence of (4.19) and (4.20). A key ingredient is the following lemma.

Lemma 4.22. *If $n \geq 0$ and A is an integer, then*

$$\sum_{j=0}^{n/2} (-1)^j \binom{n-j}{j} 4^{n-2j} \binom{2j-1}{j-A} = (-1)^A \sum_{t \geq 0} \binom{2n+2}{n-2A-4t}.$$

Proof. Thinking of either expression as a function $f(n, A)$, we establish that each satisfies

$$f(0, A) = \begin{cases} 0 & A > 0 \\ 1 & A \leq 0, \text{ even} \\ -1 & A < 0, \text{ odd,} \end{cases}$$

$f(-1, A) = 0$, and the recursive formula

$$f(n, A) = 4f(n-1, A) - f(n-2, A-1) - 2f(n-2, A) - f(n-2, A+1).$$

Thus they are equal. The recursive formula for the LHS of 4.22 is proved using several applications of (4.3), while for the RHS one proves the closely related formula

$$\binom{2n+2}{m} = 4\binom{2n}{m-1} + \binom{2n-2}{m} - 2\binom{2n-2}{m-2} + \binom{2n-2}{m-4}$$

by multiplying both sides by x^m and summing over all values of m . ■

Replace k in (4.19) by $2s + 1$, and m by $2a + 1$. Using 4.22 with $A = s + 1$, (4.19) becomes

$$\begin{aligned} & \sum_{s \geq 0} (2s + 1)^{2a+1} (-1)^{s+1} \sum_{j=0}^{n/2} (-1)^j \binom{n-j}{j} 4^{n-2j} \binom{2j-1}{j-s-1} \\ &= \sum_{j=0}^{n/2} (-1)^j \binom{n-j}{j} 4^{n-2j} \sum_{s \geq 0} (-1)^{s+1} (2s + 1) \binom{2j-1}{j-s-1} \sum_{i \geq 0} \binom{a}{i} (4s^2 + 4s)^i. \end{aligned}$$

Replacing s by $j - t - 1$ yields a formula which can easily be manipulated to (4.20), except that the i -sum is unrestricted. The restriction to $i \geq j - 1$ follows from the following lemma.

Lemma 4.23. *If $1 \leq i \leq j - 2$, then*

$$\sum_{t=0}^{j-2} (-1)^t \binom{2j-1}{t} (2j - 2t - 1) \binom{j-t}{2}^i = 0.$$

Proof. We solve the system of equations

$$\begin{aligned} a_1 \binom{3}{2} + a_2 \binom{4}{2} + \cdots + a_{j-2} \binom{j}{2} &= 1 \\ a_1 \binom{3}{2}^2 + a_2 \binom{4}{2}^2 + \cdots + a_{j-2} \binom{j}{2}^2 &= 1 \\ &\vdots \\ a_1 \binom{3}{2}^{j-2} + a_2 \binom{4}{2}^{j-2} + \cdots + a_{j-2} \binom{j}{2}^{j-2} &= 1 \end{aligned}$$

by Cramer's rule and the Vandermonde evaluation of determinants, obtaining

$$\begin{aligned} a_{s-2} &= \frac{1}{\binom{s}{2}} \frac{\prod_{t>s} \left(\binom{t}{2} - 1 \right) \prod_{t<s} \left(1 - \binom{t}{2} \right)}{\prod_{t>s} \left(\binom{t}{2} - \binom{s}{2} \right) \prod_{t<s} \left(\binom{s}{2} - \binom{t}{2} \right)} \\ &= \frac{2}{s(s-1)} \frac{\prod_{t \neq s} (t(t-1) - 2)}{\prod_{t \neq s} (t(t-1) - s(s-1))} \\ &= \frac{2}{s(s-1)} \frac{\prod_{t \neq s} (t-2)(t+1)}{\prod_{t \neq s} (t-s)(t+s-1)} \\ &= (-1)^{s+1} \frac{2s-1}{3} \frac{\binom{2j-1}{j-s}}{\binom{2j-1}{j-2}}. \end{aligned}$$

The values of t in the products always run from 3 to j . Substituting the solutions into the equations yields equations which reduce to those that were to be proved. ■

This completes the proof that the coefficient of ξ_1 in (3.20) is divisible by 2^n if $n \neq 4$. The proof that the coefficient of ξ_1 in (3.19) is also divisible by 2^n is extremely similar. Using virtually identical methods, we show it equals

$$\binom{n+1}{1} 2^{2n-3} + \sum_{j \geq 2} 2^{2n+1-4j} \left(\binom{n+2-j}{j} - \binom{n-j}{j-2} \right) \sum_{i \geq j-1} 8^i \binom{a}{i} \sum_{t=0}^{j-2} (-1)^t \binom{2j-1}{t} (2j-2t-1) \binom{j-t}{2}^i.$$

Each term of this is divisible by 2^{n+1} except when $n = 4$ and $j = 2$, in which case it is divisible by 2^n .

We complete the paper by giving deferred proofs of two lemmas.

Proof of Lemma 4.15. This lemma can probably be proved by some sort of generating function argument. However, we present a proof along the lines by which it was discovered, utilizing **Maple** and the method of [17].

As described in [17], one can often prove $f(n) := \sum_s F(n, s)$ is constant (independent of n) by finding a function G such that

$$F(n+1, s) - F(n, s) = G(n, s+1) - G(n, s). \quad (4.24)$$

If such a function G can be found, then summing this equation over all values of s yields the conclusion $f(n+1) - f(n) = 0$, and hence f is constant.

In our application, we run the software associated with the book [17] on the function

$$F(n, s) := (-1)^s \binom{n+s-1-w}{s} \binom{3n-w}{2n-s} \binom{3n-s+v}{2n} \bigg/ \binom{4n-w+v}{2n}. \quad (4.25)$$

Here n , w , and v are all free parameters, but we tell **Maple** that n is the induction parameter by running `zeil(F(n,s),s,n,N)`, where F is as in (4.25). The second argument of `zeil` is the summation variable, the third is the parameter to be used for the recursion, and the fourth is the symbol to use for the operator which increases the value of the third parameter by 1. The output in this case is `N-1`, followed by $G(n, s)$, an explicit polynomial, involving v and w in addition to n and s , for which (4.24) holds. In this case, the polynomial G involves approximately 500 monomials. The significance of the `N-1` is that this operator acting on F equals $G(n, s+1) - G(n, s)$. This says exactly (4.24). So we know that the ratio of the two expressions involved in the lemma is constant. To know that this ratio is 1, we note that when $n = 0$ each equals 1. ■

Proof of Lemma 4.18. The proof is by induction on m . The lemma is readily verified if $m = 0$. Let $s_{m,j}$ denote the sum of the two coefficients with which the lemma deals. Since $f_{m+1} = (1 - x + x^2)f_m$, we obtain

$$s_{m+1,j} = \begin{cases} s_{m,1} & \text{if } j = 0 \\ s_{m,j-1} - s_{m,j} + s_{m,j+1} & \text{if } j > 0. \end{cases}$$

The induction step follows easily. ■

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